Computers Math. Applic. Vol. 19, No. 11, pp. 29–42, 1990
Printed in Great Britain.
0097-4943/90 $3.00 + 0.00

DUAL DYNAMIC PROGRAMMING FOR LINEAR PRODUCTION/INVENTORY SYSTEMS

E. G. READ and J. A. GEORGE
Department of Operations Research, University of Canterbury, Christchurch, New Zealand

Abstract—We consider the problem of scheduling production of an inventoried item, using a production technology described by a linear program (LP). Traditional approaches involve using LP to find a schedule for a specific starting inventory, or using dynamic programming (DP) to find a production strategy for any stock level at any stage. Our method solves an LP for each period parametrically, and then uses a backwards recursion, based on the marginal conditions of DP, to generate the optimal operating strategy for the entire horizon in a convenient form. This method is dual to conventional DP in the sense that it finds optimal primal variables (points in the state space) corresponding to a set of critical shadow prices, rather than vice versa. It is more accurate and more efficient than traditional DP, while tests indicate that the computational effort required to produce a complete operating strategy is competitive with that taken to produce a single solution via LP. The method also yields useful insights into the nature of the problem. This paper concentrates on a linear deterministic problem with one inventory, but the method can be generalized. It has been successfully applied to two-dimensional reservoir release and coal stockpiling problems under uncertainty.

1. INTRODUCTION

Production/inventory systems, in which items are produced, or collected, and stored for future use in a warehouse, reservoir or stockpile, have proved to be one of the most fruitful areas for the application of Bellman's method of dynamic programming (DP). The computational challenges arising from this area have also served to stimulate many developments and refinments of the basic idea of DP. Here we develop a dual approach to DP which is particularly suited to such problems. This paper deals with the relatively simple situation in which there is only one product, with a production technology described by a linear program (LP), and we note that for this problem, our approach has strong affinities with a computational scheme suggested by Bellman himself [1]. But, our method is capable of significant generalization [e.g. 2, 3], and although it offers significant computational gains for many problems, we are also very interested in its improved accuracy, and the extra sensitivity information it provides.

Apart from DP, two approaches have been adopted to problems of this kind. Firstly, an optimal schedule can be found, for a specific starting inventory, by solving a multi-period linear or non-linear programming model, or perhaps a more efficient network technique. This can involve significant computational effort if the production model is complex, but a more important limitation in many situations is that it only provides a single production schedule and stock trajectory. Thus, a new solution is required as soon as circumstances vary sufficiently to force inventory levels away from that trajectory. This also makes it very difficult to extend such models to cover stochastic situations properly.

Secondly, the optimal strategy for some problems has a specific form, and can be found by special methods. Most such methods apply to inventory systems in a situation of stable demand, although Bahl and Zionts [4] extend the early work of Johnson [5] to provide a non-iterative solution procedure for production planning problems with one input resource and multiple products. Such methods are obviously of limited application.

DP, on the other hand, develops an optimal strategy specifying the production decision which should be made, given any stock level in any period of the planning horizon. Such a strategy may remain valid for many periods, and can be used to determine expected system performance via simulation. It also yields sensitivity analysis of a type not readily available from LP models. For instance it may reveal that a certain machine will never be used in some period, irrespective of the stock situation at the time.

But often this extra information can only be obtained at the price of a significantly greater computational burden, especially if the sub-model describing the production process in each period is complex or the state space is large. Such computational problems may prove to be a significant limiting factor, particularly when the impact of different assumptions must be explored by re-running the whole model many times.

Our method captures the advantages of DP while significantly reducing the computational requirements. For many problems it will produce a complete operating strategy with less computational effort than would be required to produce a single solution via LP. However, our method should not be seen as an approximation to DP, but rather the reverse. It constructs the optimal solution directly, whereas conventional DP only provides a discrete approximation for problems with a continuous state space. It also has the advantage of producing the optimal strategy in the form of a simple chart which can be readily understood and applied by a production manager. Many aspects of system performance can be deduced directly from this chart, while the impact of different assumptions about demand levels, discount rates, holding costs, inventory capacity, stockout costs, wastage, planning horizons or end conditions can be explored without much extra computational effort. In particular we can generate a chart summarizing the optimal "steady-stage" operating strategy, independent of any end period assumptions.

Our method may be seen as dual to conventional DP in the following sense. In DP we choose an essentially arbitrary grid of primal variables and for each one we find the optimal decision, and implied shadow price, which in this case is the marginal value of stock. Here we choose instead a special grid of dual variables, the critical marginal stock values at which the production decision changes, and for each one we find the corresponding stock level in the primal state space. The advantages of the method, in terms of accuracy and efficiency, stem from the fact that we deal directly with the critical values which determine the form of the optimal strategy, rather than trying to infer them by interpolating on an arbitrary grid. This also eliminates any need to approximate solutions by successive refinement of the grids, as is commonly done in DP. For the problem studied here the critical values of the dual variable can be determined by parametric programming on the LP describing the production technology available in each period.

Our rather geometric approach shows some affinities with early work by Dreyfus [6] on a particular inventory problem and Kaufman and Cruon [7] on "stationary policies". Computational similarities between our method and those of Bellman [1] and Nemhauser [8] are discussed later. Our duality concept appears to be a generalization of that of Ben Israel and Flam [9], who deal with a single non-renewable resource. They propose that, rather than specifying a resource stock level and then asking the question "What is the best return we can achieve over the rest of the planning horizon, given this stock level?", we could specify a level of return and ask "What is the minimum resource stock we require to achieve this return over the remainder of the planning horizon?". Labadie and Fontane [10] take a similar approach to multi-dimensional problems, but, rather than exploiting a special dual structure to characterize the whole solution strategy, they deal with problems in which there is a "g-unique" solution corresponding to each objective function value. On the other hand, Pereira and Pinto [11] apply Benders decomposition to produce locally accurate piecewise linear approximations to the marginal value functions which are central to our method.

This paper concentrates on the problem of scheduling production of a single product, which is produced in a factory whose technology is described by an LP, and then stored for future use if desired. This problem is formally equivalent to that studied by Bahl and Zionts [4], in which many products can be produced from a single inventoried resource. The special method which they develop for that problem shows some affinities with ours, but unfortunately it only appears to be applicable to the, rather unrealistic, case in which there are no upper or lower inventory limits. Thus, our method represents a significant generalization for such problems.

But, more importantly, our method is itself capable of significant generalization. Like DP it can be extended to handle uncertainty in a natural way and non-linear production technologies and/or dynamics can be handled. Multi-dimensional problems can also be modelled, although the output will not be displayed so conveniently, and computational efficiency will depend on the complexity of the decision rules required. The general approach was first applied in Ref. [2] to the problem of scheduling hydro and thermal power generation under uncertainty, in a system involving two reservoirs (inventories) over planning horizons of up to 1500 periods. In Ref. [3], the same approach has been applied to a two-dimensional stochastic DP in order to optimize coal production and stockpiling strategies.

In this paper we first introduce the basic concepts in the context of a simple model in which wastage, discounting and holding costs are ignored, while the same LP describes the production technology in each period. Under these assumptions the solution has a special form, and can be found by a particularly efficient algorithm. We then develop an algorithm for the more general case, and discuss the use of the output charts for various kinds of sensitivity analysis. Throughout, the discussion is illustrated by reference to a small example problem.

2. BASIC MODEL

Consider the linear production planning model for a single product with multiple input resources and multiple processes. We can express the problem as:

minimize

$$z = \sum_t [(C_t)'\mathbf{X}_t + c_t y_t] - v_T(s_T)$$

subject to

$$(A_t)'\mathbf{X}_t + y_t = B_t \qquad t = 1, \ldots, T$$

$$S_{t-1} + y_t - s_t = d_t \qquad t = 1, \ldots, T$$

$$SMIN_t \leqslant s_t \leqslant SMAX_t \qquad t = 1, \ldots, T, \tag{1}$$

where

$\mathbf{X}_t$—a vector of activity variables for inputs, intermediate commodities, processes etc. in period t,
y_t—output of the final product in period t,
s_t—stock of the final product at the end of period t,
d_t—the demand for the final product in period t

and

$v_T(s_T)$—the value of the end-of-horizon inventory.

Although this model will later be generalized to include various kinds of inventory holding cost, it is quite important in its own right, particularly in the area of short- to medium-term water release scheduling for reservoirs. In particular it includes the trivial case for which the solution of the "LP" merely consists of ranking alternative water uses, or alternative sources of supply, in ascending order of cost.

We do not specify an opening stock level because our solution method produces an operating strategy valid for all opening stock levels. At the end of the planning horizon v_T ensures that stocks are maintained at a reasonable level. We require v to be concave, so that the marginal value curve $m_T(s_T)$ is monotonically decreasing as in Fig. 1(a) (i.e. lower stock levels imply higher values for additional stock). A final target can be represented by setting the marginal value of stock to zero above the target level and to some penalty value below it. Although our method will handle any concave v_T function, our example will use a piecewise linear function, so as to facilitate comparison with LP.

Instead of solving this problem directly, we solve each single period sub-problem parametrically to produce a "supply function" for that period, as demonstrated in the next section. In later sections we will use the principles of DP to develop a backward recursion which combines these supply functions with the end-of-period value functions to define an optimal management strategy.

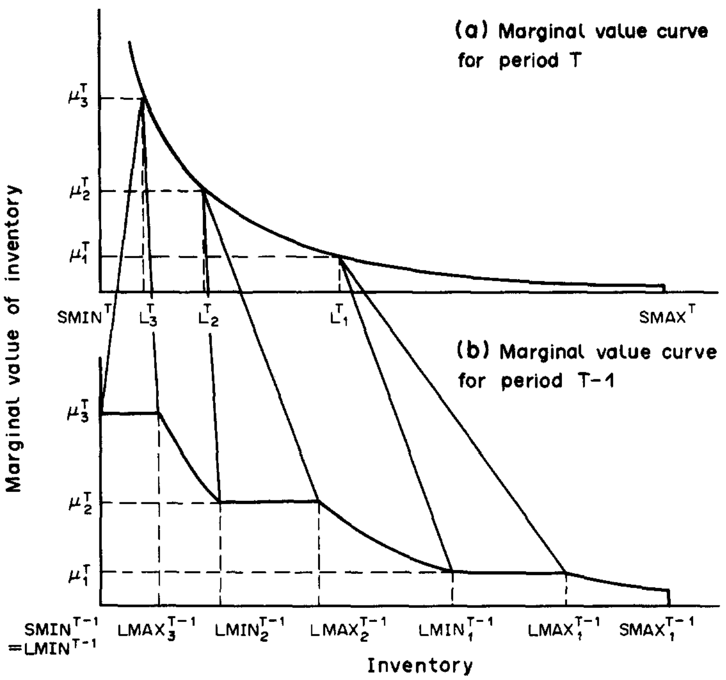

Fig. 1. Inventory marginal value curves for periods T and $T-1$.

3. SINGLE-PERIOD SUPPLY CURVES VIA PARAMETRIC PROGRAMMING

The solution of the following single-period problem determines the lowest cost at which d_t units can be produced by the production technology available in t:

minimize

$$z_t = (C_t)'\mathbf{X}_t$$

subject to

$$(A_t)'\mathbf{X}_t + y_t = B_t$$
$$y_t = d_t. \tag{2}$$

First we must solve problem (2) parametrically for d_t ranging within the predetermined lower and upper limits of output for the product for each $t = 1, \ldots, T$. This yields a series of optimal basic solutions, $i = 1, \ldots, I(t)$, over the range of feasible output of the final product in t. These define a series of distinct "production processes", with the marginal cost of production for process i being μ_{it}, the dual value of the constraint $y_t = d_t$ at the ith basis. Thus, the "supply curve" for the product in each period, consists of a series of steps, with the cost of production being μ_{it} over the output range from Q_{i-1t} to Q_{it}. We define the last process to have $Q_{It} = \infty$ and μ_{It} to be a large penalty value.

A trivial example of this kind of supply curve is provided by the "merit order" of thermal stations in a hydro/thermal power system, as in Ref. [2]. But we will illustrate our technique using a simple model with four production processes $x1$ to $x4$ on four machines S, T, U and V using regular time and overtime, as represented by the following LP for each period. For simplicity we will assume the model is identical for each period:

minimize

$$4.98x1 + 5.6x2 + 5.7x3 + 5.62x4 + 0.2S\mathrm{OT} + 0.175T\mathrm{OT}\ 0.12U\mathrm{OT} + 0.15V\mathrm{OT}$$

subject to

$$6x1 + 8x2 - SOT \leqslant 480$$
$$6x3 + 5x4 - TOT \leqslant 480$$
$$7x1 + 10x2 + 3x4 - UOT \leqslant 480$$
$$3x1 + 9x3 + 8x4 - VOT \leqslant 480$$
$$x1 + x2 + x3 + x4 - y = 0$$
$$y = d$$
$$SOT \leqslant 60, \quad TOT \leqslant 60, \quad UOT \leqslant 60, \quad VOT \leqslant 60.$$

Parametric programming on the constraint $y = d$ yields the set of bases in Table 1. (The solution values have been rounded.) So the supply curve for each period is given by:

$$Q_i = 0, 70, 100, 105, 115, 500$$

and

$$\mu_i = 0, 5, 5.7, 6, 7, 25, 200.$$

4. BASIC METHOD

The key to our method is the efficient generation of functions, $m_t(s_t)$, defining the marginal value of stock at the end of each period. Provided m_t is a monotone function of s_t, we may define upper and lower "guidelines" for each production process by:

$$LMAX_{it} = \max\{s_t : m_t(s_t) \geqslant \mu_i\},$$
$$LMIN_{it} = \min\{s_t : m_t(s_t) \geqslant \mu_i\}. \tag{3}$$

For ease of exposition we will first consider the problem faced in the final period, T, and assume that m_T is strictly monotone, as in Fig. 1(a), so that there is only one guideline for each process:

$$L_{iT} = LMAX_{iT} = LMIN_{iT}. \tag{4}$$

Our method is based on the following simple propositions.

Proposition 1

The optimal solution involves equating the marginal cost of production in each period with the marginal value of inventory at the end of that period.

This is intuitively obvious, but can also be derived from the optimality conditions of the problem. In particular, if process i is partially utilized in T, so that μ_i is the marginal production cost, the marginal value of stock at the end of the period must also equal μ_i. That is, $s_T = L_{iT}$. Conversely, if $s_T = L_{iT}$, then the marginal production cost in T can not be either higher or lower than μ_i. Thus, we have:

Proposition 2

The optimal stock level at the end of T will be L_{iT} if, and only if, production process i is fully or partially utilized in T, so that the marginal production cost is μ_i.

Table 1. Basis changes for parametric programming of d

	Basis						
	1	2	3	4	5	6	7
$x1$	0	70	70	77	77	0	NA
$x2$	0	0	0	0	0	54	NA
$x3$	0	0	30	28	33	60	NA
$x4$	0	0	0	0	0	0	NA
y	0	70	100	105	110	115	500
μ	0	5	5.7	6	7	25	200

Further, since only a limited range of production is possible at a marginal cost of μ_i, there can only be a limited range of stock levels, at the beginning of period T from which it is optimal to aim at L_{iT}. In fact:

Proposition 3

(a) It will be optimal to fully utilize process i in T if, and only if, the opening stock level for T lies on or below

$$X\text{MIN}_{iT-1} = L_{iT} + d_T - Q_{iT}. \tag{5}$$

(b) It will not be optimal to utilize process i at all if, and only if, the opening stock level for T lies on or above

$$X\text{MAX}_{iT-1} = L_{iT} + d_T - Q_{i-1T}. \tag{6}$$

The first result follows from the fact that if $s_{T-1} = \text{XMIN}$, the highest end-of-period inventory level we could achieve by fully utilizing process i, would be

$$s_T = X\text{MIN}_{iT-1} - d_T + Q_{iT} = L_{iT}. \tag{7}$$

Thus if s_{T-1} lies below **XMIN**, and we were to only utilize process i, the end-of-period stock would fall below L_{iT}, and its marginal value must therefore exceed μ_i, implying that full utilization of i, and possibly of more expensive processes, is justified. On the other hand, if s_{T-1} lies above **XMIN**, full utilization of i would raise final stocks above L_{iT}, implying a marginal stock value below μ_i. Thus, by Proposition 1, full utilization of i could not be justified.

The second result follows similarly, since if s_{T-1} lies above **XMAX** the end-of-period inventory must exceed L_{iT} even if we adopt the minimum production level possible with process i, Q_{i-1T}.

Now, combining Propositions 2 and 3 yields:

Proposition 4

If opening stock lies between $X\text{MIN}_{iT}$ and $X\text{MAX}_{iT}$, the optimal decision is to partially utilize process i, aiming for a final inventory of L_{iT}. If it lies between $X\text{MIN}_{iT}$ and $X\text{MAX}_{i+1T}$, it will be optimal to fully utilize i, but not utilize $i+1$ at all.

In fact it can be seen that, if s_T lies between $X\text{MIN}_{iT}$ and $X\text{MAX}_{iT}$ the optimal utilization of process i may be determined by linear interpolation between Q_{i-1T} and Q_{iT}. Thus if s_T lies half way between $X\text{MIN}_{iT}$ and $X\text{MAX}_{iT}$, then half utilization of i is optimal.

Now we may utilize Proposition 1 again to define the marginal stock value curve for the beginning of period T/end of period $T-1$:

Proposition 5

The marginal value of the inventory at the end of period $T-1$ is given by:

$$\begin{aligned} m_{t-1}(s) &= \mu_i && \text{if } X\text{MIN}_{it-1} \leqslant s \leqslant X\text{MAX}_{it-1}, \\ m_{t-1}(s) &= m_t(s - d_t + Q_{it}) && \text{if } X\text{MAX}_{it-1} \leqslant s \leqslant X\text{MIN}_{i-1t-1}. \end{aligned} \tag{8}$$

Figure 1(b) demonstrates the formation of the marginal value curve for the period $T-1$ from the final marginal value curve. "Flats" have been inserted into the curve for each production process, with a constant marginal value for stock between $X\text{MIN}_{iT-1}$ and $X\text{MAX}_{iT-1}$. Between the flats the curve is identical to the sections of the original curve between the corresponding L_{iT} values. Clearly m_{T-1} will be monotone, provided m_T is. The marginal value at $S\text{MIN}_T$ is infinite and at $S\text{MAX}_T$ is zero. Thus, we can immediately provide a characterization of the guideline levels for the end of period $T-1$, by:

Proposition 6

$$\begin{aligned} L\text{MIN}_{iT-1} &= \max\{\min\{X\text{MAX}_{iT}, S\text{MAX}\}, S\text{MIN}\}, \\ L\text{MAX}_{iT-1} &= \max\{\min\{X\text{MIN}_{iT}, S\text{MAX}\}, S\text{MIN}\}, \end{aligned} \tag{9}$$

Now the reasoning we have outlined can clearly be replicated for all periods to produce guidelines by a DP-style backwards recursion. The only change required is to note that, in general, $L\mathrm{MAX} > L\mathrm{MIN}$, so that equations (5) and (6) above must be modified to

$$X\mathrm{MAX}_{it-1} = L\mathrm{MAX}_{it} + d_t - Q_{i-1t},$$
$$X\mathrm{MIN}_{it-1} = L\mathrm{MIN}_{it} + d_t - Q_{it}. \tag{10}$$

Thus, for this implified problem we have:

Algorithm 1

(A) Solve the single-period sub-problem parametrically to produce a supply curve.

(B) Apply equations (10) and (9) recursively to produce guidelines for each period in the planning horizon.

These guidelines, which completely characterize the optimal strategy, can be conveniently displayed on a simple chart, as in the example below. It should be observed that for this basic case the shape of the end-of-horizon marginal value function between the guideline levels is largely irrelevant. Provided it is monotone so that we can identify guidelines from the curve, the algorithm will produce optimal management guidelines for the entire period, without ever using equations (8) to update the marginal value curves. In fact, each guideline can be produced quite separately from the others, a potentially useful way of producing approximate solutions for large problems. Unfortunately, these observations will not be valid for the more general case considered below.

It may also be observed that, as we work backwards, the marginal value curves become increasingly dominated by the flats corresponding to each production process, which become wider and wider while the segments of the original value curve are forced out of the feasible region. This effect, which may be observed in the example problem, is also peculiar to the basic model.

5. SOLUTION OF THE EXAMPLE PROBLEM

The example in Section 3 is solved here, assuming a 12-month planning cycle with the following demands for $t = 1$ to 12:

$$d_t = 110,\ 120,\ 110,\ 90,\ 80,\ 60,\ 100,\ 160,\ 70,\ 70,\ 40,\ 100.$$

For simplicity we assume that the limits on stock levels are constant at: $S\mathrm{MAX} = 200$ and $S\mathrm{MIN} = 0$. The value of stock at the end of T (v_T) is assumed to be a piecewise linear function. Thus the marginal value function $(m_T(s))$ is a staircase function assumed to be as follows:

$$s = 200,\ 180,\ 130,\ 120,\ 100,\ 80,\ 60,\ 40,\ 20,\ 0$$

$$m_T(s) = 1,\ 3,\ 5.5,\ 5.9,\ 6.5,\ 9,\ 12,\ 18,\ 30,\ 500.$$

Figure 2 shows how the flats, delimited by the guidelines, are inserted to form the marginal value curve for the end of period $T-1$. Continuing this process yields the solution displayed as a chart of production guidelines in Fig. 3. The flats in the marginal value curve correspond to the "fans" numbered in the production chart. In general we would expect these to show stock values from $L\mathrm{MIN}_{it-1}$ to $L\mathrm{MAX}_{it-1}$ in period $t-1$, being mapped to the same final stock level, L_{it}, as in period T. But because the same production cost levels occur in each period, and we have ignored inventory holding costs and discounting, the fans spread as we work back from period T. Each new fan is amalgamated with the corresponding old fan, because it has the same marginal cost. Conversely, the fans for some production processes do not appear at all because they are outside the feasible inventory range.

The top line of each fan corresponds to a zero use of that process (or full use of the previous process) and the bottom line of the fan to the full use of the process. For stock levels between the fans, full use of the process for the fan above is optimal. Within fans partial use of the corresponding process is optimal, although, for this case, there are many alternative optima. Clearly, if $L\mathrm{MAX}_i = L\mathrm{MIN}_i = S\mathrm{MAX}$ or $S\mathrm{MIN}$ then process i should not be used, but if a fan

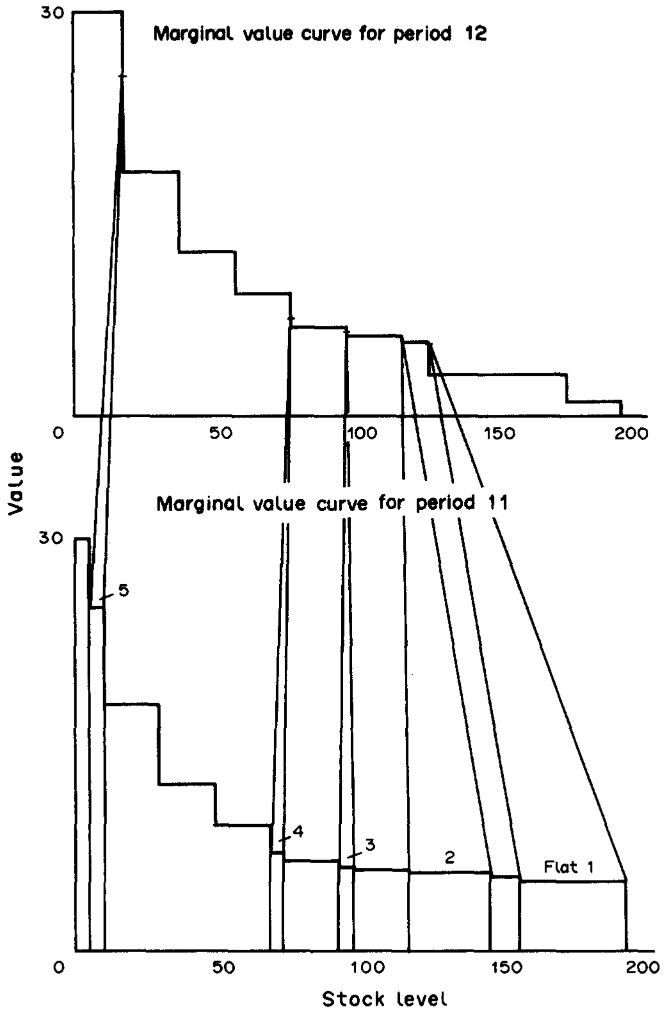

Fig. 2. Inventory marginal value curves for periods T and $T-1$.

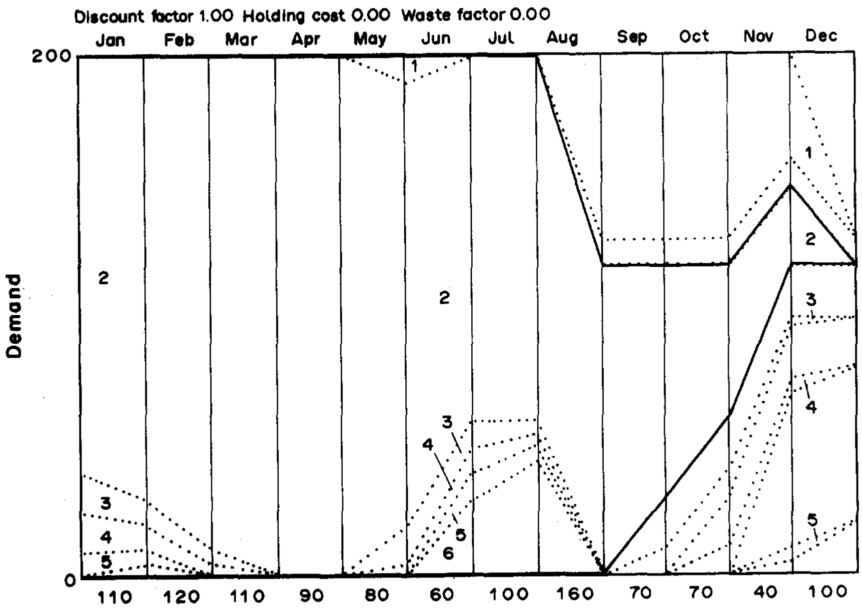

Fig. 3. Production guideline chart.

has been truncated by an inventory capacity bound, say $SMAX$, care should be taken to interpolate between XMIN($= LMIN$) and XMAX′, given by

$$XMAX'_{it} = LMIN_{it} + Q_{it} - Q_{i-1,t}. \tag{11}$$

Thus, an optimal trajectory of production and inventory can be traced from any stock level in any period to the end of the planning horizon. In Fig. 3, for any starting stock position between the dark lines, the optimal trajectory leads to the same final stock level (120 units).

The advantage of this chart over an LP solution is that it displays all of the optimal strategies. Not only does it allow the solution to be read for a wide range of situations, but it also yields insights which are not available from LP sensitivity analysis. For instance, in this case, we can immediately see that process 2 will dominate production over most of the period, unless stock levels become quite extreme. Also, the actual utilization level of process 2 is not important as long as the inventory is within the bounds of the fan for process 2. However, the areas at the bottom of the chart use process 6—the artificial process. At those inventory levels there is no feasible production level. Management must seek to avoid this zone.

The structure of the chart does not depend on the complexity of the method used to generate the Q_i and μ_i. Moreover, we can alter the demand levels, inventory capacity or the value of the inventory in the final period, without re-running the production model, since the supply curve does not change.

We can also use this data to establish a steady-state solution, assuming that the production model remains valid, while the seasonal demand pattern is typical. Having run the model to produce a marginal value curve for the beginning of period 1, we re-run it, with this curve replacing the marginal value curve for period T, continuing in this way until a stable chart is produced. For this model, the steady-state chart shown in Fig. 4 was found after two cycles. This chart reveals that, in equilibrium, production is not only dominated by process 2, but there are optimal trajectories, such as ACDE, which only require a fraction of the available inventory capacity.

Although this result may appear pathological, the problem is not due to the solution method, but to the assumptions of the model. If no costs of any kind are associated with holding stock, then any deterministic optimization will yield the same solutions. Our method simply makes the pathological consequences of this obvious, and thus raises important questions about the, relatively common, practice of applying this type of model to reservoir release scheduling, for example.

The steady-state chart also implies the true marginal value curves for stock under steady-state assumptions. In general these will be dominated by the flats inserted by the method, with any

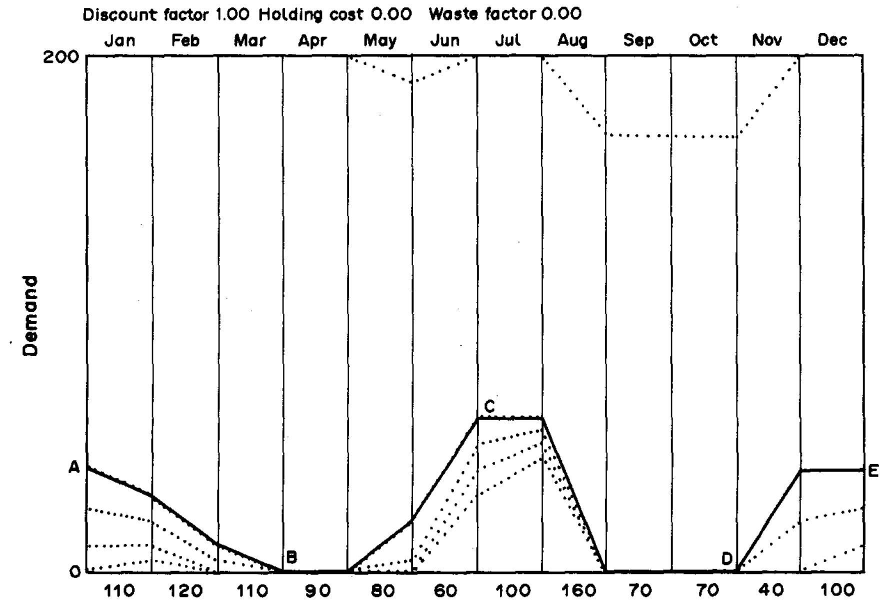

Fig. 4. Steady-state chart.

residual sections of the original marginal vlaue curve for T not affecting the solution in any way. Thus, we can use an arbitrary marginal value function to start the process.

Finally, we should note that this approach can be extended to handle the situation in which there is a different LP describing the production options available in each period. If the marginal production costs do not vary, or the number of alternative production levels is small, a slightly modified version of the above method is suitable. Otherwise, the method of the next section would be more appropriate.

6. HOLDING COSTS, WASTAGE AND DISCOUNTING

We now extend the model to include three types of inventory holding "cost"; direct holding costs for the inventory, wastage of goods in the inventory and discounting of revenue and costs. We also allow the production options available in each period to be described by a different LP. So μ_{it} is the marginal cost of the ith production process in t. We also assume the following:

h_t—the holding cost per unit of inventory in period t,
α—the discount factor per period

and

w_t—the wastage factor for period t.

Problem (1) now becomes:

minimize

$$z = \sum_t \alpha^t[(C_t)'\mathbf{X}_t + c_t y_t - h_t s_t] - \alpha^T v_T)s_T)$$

subject to

$$\begin{aligned} (A_t)'\mathbf{X}_t + y_t &= B_t & \mathrm{t} &= 1, \ldots, \mathrm{T} \\ (1 - w_t)s_{t-1} + y_t - s_t &= d_t & t &= 1, \ldots, T \\ S\mathrm{MIN}_t \leqslant s_t &\leqslant S\mathrm{MAX}_t & t &= 1, \ldots, T. \end{aligned} \tag{12}$$

This change preserves the basic structure of the problem, and the same basic algorithm applies, with some modification to the formulae. The new relationships can be established easily from the optimality conditions of the new sub-problems, but they also follow intuitively.

Proposition 1 must be adapted because the effect of discounting, holding costs and wastage is to increase the effective marginal cost of production relative to the value of the inventory at the end of the period. We now wish to identify the end-of-period stock level at which the marginal value of inventory equals the net marginal cost of production from process i, after these effects have been accounted for. If we let m_t be the value of stock at the end of t, before period t holding costs and period $t + 1$ wastage, the appropriate guideline level is defined by

$$m_t(L_{it}) = \mu'_{it} = (\mu_{it}/\alpha) + h_t. \tag{13}$$

In this context the guideline level will generally be unique, since there is no longer any reason for μ'_{it} to coincide with the marginal value of any of the flats in the marginal stock value curve.

Having identified L_{it}, the formulae for updating $\mathbf{XMAX}_t$ and $\mathbf{XMIN}_t$ given in equations (5) and (6) change, because of the effect of wastage, to

$$\begin{aligned} X\mathrm{MIN}_{it-1} &= (L_{it} + d_t - Q_{i-1t})/(1 - w_t) \\ X\mathrm{MAX}_{it-1} &= (L_{it} + d_t - Q_{it})/(1 - w_t). \end{aligned} \tag{14}$$

XMAX and **XMIN** are still interpreted as in Propositions 3 and 4, while the marginal stock value curve for the beginning of a period is constructed using Proposition 5 with the formulae adapted for the adjustments in equation (13). Thus, the marginal value curve for $t - 1$ not only has new flats, but is also transposed additively and/or multiplicatively from that for period t, via

$$\begin{aligned} &m_{t-1}(s) = (1 - w_t) * \mu_{it} \quad \text{if } L\mathrm{MIN}_{it-1} \leqslant s \leqslant L\mathrm{MAX}_{it-1} \\ &m_{t-1}(s) = \alpha * (1 - w_t) * (m_t((1 - w_t)s - d_t + Q_{it}) - h_t) \quad \text{if } L\mathrm{MAX}_{i+1,t-1} \leqslant s \leqslant L\mathrm{MIN}_{i,t-1}. \end{aligned} \tag{15}$$

These changes significantly affect the solution algorithm. Proposition 6 no longer holds, even if the LP for each period is the same. Thus, the end-of-period guideline levels are calculated using equations (3) and we must now explicitly update the marginal value curves for each period.

The nature of the optimal strategy also changes significantly. Previously the flats in the marginal value curves grew as we worked back from period T. The inclusion of any of these additional factors destroys this effect. So, in general, a series of unique new flats will be created at each stage, there will be few, if any, alternative optimal solutions and the steady-state trajectory will be unique. We are no longer indifferent between producing at the same marginal cost in different periods, preferring to defer production whenever possible. Thus, we would expect the guidelines to be consistently lower than for the basic model.

Thus for the general case we have:

Algorithm 2

For each period $t = T, \ldots, 0$:

(A) Produce a supply curve by solving its LP model parametrically.

(B) Identify guidelines from the end-of-period value curve via equations (3).

(C) Insert flats and update the marginal value curve via equations (14) and (15).

This algorithm defines all of the marginal value curves and associated decisions exactly at each stage. Thus, it is actually more accurate than the discrete approximation produced by conventional DP, although this inaccuracy will not always be significant in practice.

We also claim that this method is more efficient, in the sense that conventional DP requires greater computational effort to produce a piecewise linear approximation to the stock value curve with the same number of segments as the exact curve produced by our method. This is because, instead of testing several production levels to determine the optimum for each stock level, we generate all of the solutions required in a single parametric run. We then apply the stock balance equation (12), and adjust m via equations (15), only once for each step on the new curve.

The only evident inefficiency in our method is that it may produce some LP solutions which are never used because the corresponding flats are out of bounds. This may be overcome by integrating the LP and dual DP phases and thus performing only those basis changes which are required during a single pass through the marginal value curve. In fact we could form a single-period LP for each successive period using the piecewise linear representation of the value of the inventory at the end of the perod generated by backward recursion to that time. Varying s_t parametrically will then generate the marginal value curve for the beginning of the period. Bellman [1] and Nemhauser [8] applied methods of this type in a forward direction to produce a single solution to time-staged LPs. More recently, Scott [12] has proposed a decomposition method which builds up an approximation to the marginal value curves as it converges to a single optimal solution, and a similar method is used to optimize reservoir release strategies in Ref. [11].

7. ANALYSIS OF THE GENERAL PROBLEM

Consider again the example in Section 3. For each month assume a discount factor of 0.99, wastage of 0.1 and a holding cost of $0.1.

Figure 5 shows the production chart for this problem. As expected, rather than expand existing production fans in each period, the method creates new ones. The fans are still interpreted in the same way, but the range of inventory levels between the fans, for which some process is run at full capacity, is now much greater. In these sections between the fans the marginal value curve may have a number of steps, most of which correspond to the use of some production process in some future period. Note that the presence of various kinds of holding cost provides a considerable disincentive for early production, so that much less effort is expended to keep stock levels up in this chart. Hence process 1 becomes optimal in many situations, and process 2 is not employed until stocks are quite low.

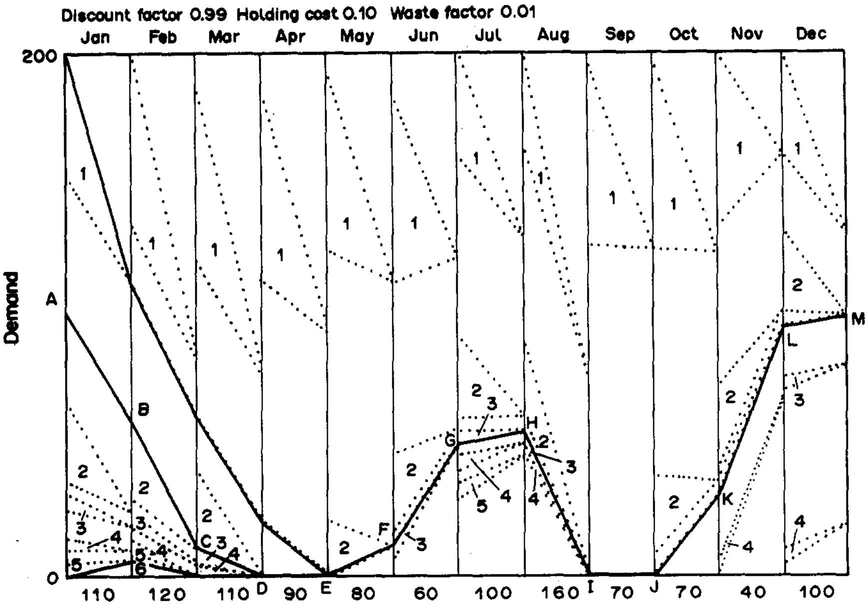

Fig. 5. Production chart for the problem using discounting, wastage and holding costs.

We can use this production chart to manage the system, and also to perform a variety of analyses.

(i) The optimal production schedule to the end of the planning horizon, starting from any inventory level in any period, can be found by tracing the trajectory of optimal inventory levels at the beginning of each period. For example, the optimal trajectory, starting from an inventory level of 110 on 1 January, is shown by the line A, B, C, . . . , M in Fig. 5.

(ii) Expected costs, inventory levels etc. can be determined, by simulation using the operating rules shown on the chart. (This, in fact, is the major purpose of the hydro scheduling model reported in Ref. [2].) These estimates are valid on the assumption that management will in fact operate in this way, i.e. that they will use an LP model to plan production, for this simple example.

(iii) The impact of certain processes not being available can be ascertained at a glance. In this case, for instance, process 1 is the only one which could possibly be required in September.

(iv) The sensitivity of production schedules to the starting inventory is easily determined. Figure 5 shows the trajectories for inventory levels of $SMAX$ and $SMIN$ at the beginning of January (shown as solid lines starting at each of those points). These trajectories converge towards each other quite quickly so that by April they coincide. Thus, for this problem, the optimal strategy for most of the planning period is independent of the starting inventory level.

(v) The steady-state optimal strategy shown in Fig. 6, can be found iteratively, as for the simple model. If the ending inventory level found in analysis (iv) (point M in Fig. 5) is used as the beginning inventory for the next iteration, the resultant trajectory, UVWXU, on the steady-state production chart gives us the optimal steady-state production/inventory schedule. It should be noted that it now becomes profitable to incur the extra cost of utilizing process 3 at some times of the year, rather than always using only process 2, so as to avoid inventory holding costs.

(vi) To manage the system during a transitional period, the marginal value curve for the end of the period of expected change should first be determined from the steady-state chart for the situation after the transition. The method can then be applied, working backwards from that time, to define optimal management strategies over the transitional period.

(vii) Planning horizons and planning cycles can also be determined. In the steady-state chart the optimal trajectory for any initial inventory level will eventually converge to the steady-state trajectory. This fact can be used to determine planning cycles. In Fig. 6 any trajectory with an initial inventory level below A will converge to V, and any starting from a level above A to X. (The trajectories have been projected *back* from points V and X, in a manner analogous to the derivaiton of the optimal trajectories.) As long as the inventory lies below the line AV, it is optimal to aim for zero inventory at the end of April. When the inventory rises above AV, we should plan for zero inventory at the end of August. When the inventory rises above the BX line, we should plan towards next April. So we have two planning cycles.

(vii) It is easy to determine the impact of any changes to

—the demand for the product in each period,
—the inventory value curve in period T,
—the inventory bounds,
—the inventory holding costs,
—the wastage factor,
—the discount factor

or

—additional supply or disposal options not included in the LP.

In all of these cases it is only necessary to recreate the chart, which takes a few seconds on a microcomputer. Thus, we can quickly and easily see the way in which the whole management strategy should change in response to a wide variety of alternative scenarios with little computational effort. This goes a considerable way towards meeting a common criticism of mathematical programming methods, that the solutions provided are too stark and fail to give the decision-maker an understanding of the context of the optimum.

8. COMPUTATIONAL ASPECTS

In Section 6 we compared our method with DP in theoretical terms, concluding that it is more efficient and more accurate. We have also shown that, while it is no more accurate than LP, it generates considerably more information. We now wish to compare it with LP in terms of computational efficiency. Our comparison is based on the 12-period version of the example problem, direct solution of which took 120 s using a Pascal program, TURBOLP, on an IBM PC. A further 20 s were required to generate a standard sensitivity analysis.

Because TURBOLP does not have facilities for formal parametric programming, the supply curve in Section 3 was generated in a series of separate runs of the 1-period problem. But we conservatively estimate a computational time of 2 s to perform the five basis changes required. To form the chart for the basic problem in Section 5, using a compiled BASIC program of Algorithm 2, only took about 1 s because there are only a few flats in this problem. (Algorithm 1 would be even faster for this case.) Thus, a total of about 3 s is required to solve the basic problem by our method.

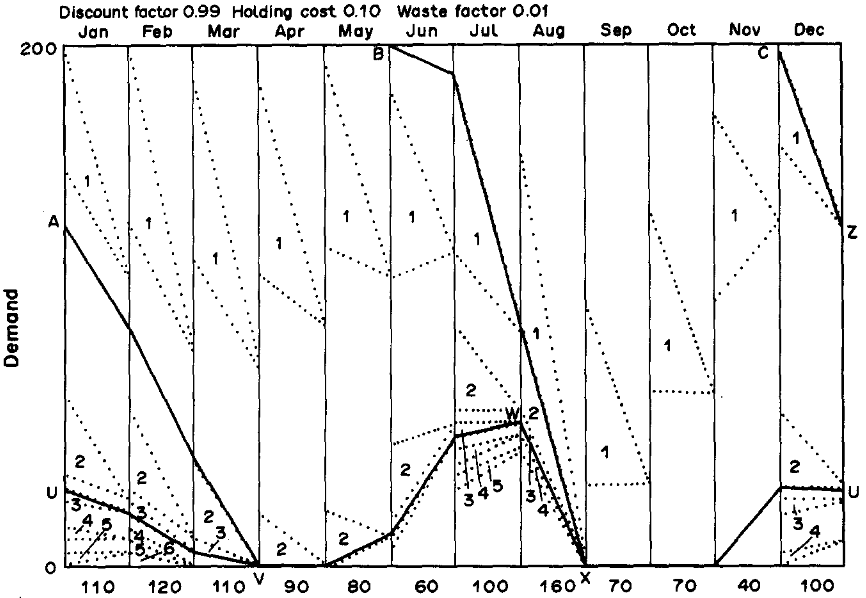

Fig. 6. Steady-state chart and production strategy under discounting, wastage and holding costs.

The problem of Section 8 (with many more flats) required about 5 s to form the chart, giving a total of about 7 s. If a different LP has to be solved for each period, computation times will obviously increase, reaching about 30 s if 12 separate LPs have to be solved from scratch. This is still only 25% of the time for the LP to form a single solution. A further 10 s was required for two more runs of the BASIC program to form the steady-state chart.

Our method is clearly competitive with LP for this type of problem. We would expect it to be even more efficient for large problems which yield supply curves with comparatively few steps, and for those in which the flats are large relative to the inventory capacity. However, we do not claim that it is the most efficient method to generate a *single solution* to this type of problem. Indeed our approach suggests some network and decomposition analogues which should be even efficient, although generating less information.

9. CONCLUSION

We have developed a hybrid technique, using DP and LP, to schedule production of a single product over a planning horizon. The method handles limited inventory capacity, discounting, wastage and holding costs, and will accommodate any concave value function for ending the inventory. It only requires the solution of single period optimization models, but generates an optimal management strategy for all periods in the form of a production chart.

This chart is not only a practical management tool, but also reveals much information about the system. Little computational effort is required to perform extensive post-optimal analysis on a variety of factors, or to determine an optimal steady-state policy.

Although we have only considered a deterministic linear problem with a one-dimensional state space, this approach is capable of significant generalization. Like DP it can be extended to handle uncertainty in a natural way [13]. Multi-dimensional problems can also be modelled, although computational efficiency will depend on the complexity of the decision rules generated. The approach has already been successfully applied to two large real-life problems, formulated as two-dimensional stochastic DPs [2, 3]. Non-linear production technologies and/or dynamics can also be handled, provided the functions involved are differentiable.

REFERENCES

1. R. E. Bellman, On the computational solution of linear programs involving almost block diagonal matrices. *Mgmt Sci.* **3,** 403–406 (1957).
2. E. G. Read, A dual approach to stochastic dynamic programming for reservoir scheduling. In *Dynamic Programming for Optimal Water Resources Systems Analysis* (Ed. A. O. Esogbue), pp. 361–372. Prentice-Hall, Englewood Cliffs, N.J. (1989).
3. P. D. Casseboom and E. G. Read, Dual dynamic programming for coal stockpiling. *Proc. O.R. Soc. N.Z.*, pp. 15–18 (1987).
4. H. C. Bahl and S. Zionts, A non-iterative multi-product multi-period production planning method. *Ops Res. Lett.* **1,** 219–221 (1982).
5. S. M. Johnson, Sequential production planning over time at minimum costs. *Mgmt Sci.* **3,** 435–437 (1957).
6. S. E. Dreyfus, An analytical solution of the warehouse problem. *Mgmt Sci.* **4,** 99–104 (1957).
7. A. Kaufman and R. Cruon, *Dynamic Programming.* Academic Press, New York (1967).
8. G. L. Nemhauser, Decomposition of linear programs by dynamic proramming. *Nav. Res. Logist. Q.* **11,** 191–196 (1964).
9. A. Ben Israel and S. D. Flam, Input optimization for infinite horizon discounted programs. *J. Optimizn Theory Applic.* **61,** 347–357 (1989).
10. J. W. Labadie and D. G. Fontane, Objective space dynamic programming approach to multi-dimensional problems in water resources. In *Dynamic Programming for Optimal Water Resources Systems Analysis* (Ed. A. O. Esogbue), pp. 88–115. Prentice-Hall, Englewood Cliffs, N.J. (1989).
11. M. V. F. Pereira and M. V. G. Pinto, Stochastic optimisation of a multireservoir hydroelectric system: a decomposition approach. *Wat. Resour. Res.* **21,** 779–792 (1985).
12. D. M. Scott, A dynamic programming approach to time staged convex programs. Technical Report SOL 85-3, Systems Optimization Lab., Stanford Univ. Calif. (1985).
13. E. G. Read and J. A. George, Stochastic dual dynamic programming for production/inventory systems (in preparation).